CARTESIAN PRODUCTS AND FUNCTIONS

DEFINITION: Given sets X and Y with $x \in X$ and $y \in Y$, we define the **ordered pair** $(x, y) = \{\{x\}, \{x, y\}\}$.

It follows from the definition (there's some work here!) that $(a, b) = (c, d)$ iff $a = c$ and $b = d$.

DEFINITION: If X and Y are sets, the **Cartesian product** of $X \times Y = \{(x, y) : x \in X \text{ and } y \in Y\}$.

EXAMPLE: Let $X = \{1, 2\}$ and $Y = \{a, b, c\}$. Find the following Cartesian products:

- $X \times Y =$
- $Y \times X =$
- $X \times X =$

EXAMPLE: Let $X = [1, 2]$ and $Y = [3, 4] \cup [5, 6]$. Sketch the following Cartesian products in the plane:

- $X \times Y$
- $Y \times X$
- $Y \times Y$

NOTE: The xy-plane is often written as $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$

PROPERTIES OF $\times$**:** Suppose $A, B \subseteq X$ and $C, D \subseteq Y$.

- $A \times \emptyset = \emptyset \times A = \emptyset$.

- $(A \cap B) \times C = (A \times C) \cap (B \times C)$

- $(A \cap B) \times (C \cap D) = (A \times C) \cap (B \times D)$

QUESTION: How does $\times$ work with $\cup$? That is, what is the relationship, in any, between the sets:

- $(A \cup B) \times C$ and $(A \times C) \cup (B \times C)$?

- $(A \times C) \cup (B \times D)$ and $(A \cup B) \times (C \cup D)$?

DEFINITION: A **relation** R from X to Y is a set $R \subseteq X \times Y$. If $X = Y$, we say R is a relation on X.

A **function** F from X to Y is a relation such that for all $x \in X$ there is a **unique** $y \in Y$ with $(x, y) \in F$.

If F is a function from X to Y we write $F : X \to Y$ and if $(x, y) \in F$, we write $y = F(x)$.

EXAMPLE: Let $X = \{1, 2\}$ and $Y = \{a, b, c\}$. Which of the following are functions from X to Y?

- $\{(1, a), (2, b)\}$
- $\{(1, a)\}$
- $\{(1, a), (2, a)\}$
- $\{(1, a), (2, a), (2, b)\}$

EXAMPLE: Let $F = \{(x^2, x) : x \in X\}$. For which sets X below is F a function on X?

- $X = \mathbb{R}$
- $X = \mathbb{R}^+ = (0, \infty)$
- $X = \mathbb{N}$

DEFINITION: Suppose $A \subseteq X$ and $C \subseteq Y$ and $F : X \to Y$.

- The **domain** of F is X and the **codomain** of F is Y.
- The **image** of A under F is the set $F(A) = \{F(a) : a \in A\}$.

  NOTE: $y \in F(A)$ iff there exists $a \in A$ with $y = F(a)$.
- The **range** of F is the image of X: $F(X)$.
- The **pre-image** of C under F is the set $F^{-1}(C) = \{x \in X : F(x) \in C\}$

  NOTE 1: $x \in F^{-1}(C)$ iff $F(x) \in C$.

  NOTE 2: Since F is a function from X to Y, it follows that $F^{-1}(Y) = X$. (Do you see why?)

EXAMPLE: Let $F : \mathbb{R} \to \mathbb{R}$ be defined by $F(x) = |x|$. Find the following:

- $F([-2, 1]) =$
- the range of $F =$
- $F^{-1}([1, 2]) =$

INDUCED SET FUNCTIONS (IMAGES): Suppose $A, B \subseteq X$, $C, D \subseteq Y$ and $F : X \to Y$.

- If $A \subseteq B$, then $F(A) \subseteq F(B)$.

- $F(A \cup B) = F(A) \cup F(B)$. More generally, $F\left(\bigcup_{\alpha \in \Delta} A_\alpha\right) = \bigcup_{\alpha \in \Delta} F(A_\alpha)$

- $F(A \cap B) \subseteq F(A) \cap F(B)$. More generally, $F\left(\bigcap_{\alpha \in \Delta} A_\alpha\right) \subseteq \bigcap_{\alpha \in \Delta} F(A_\alpha)$

  NOTE: Examples exist where $F(A \cap B) \neq F(A) \cap F(B)$.

- $A \subseteq F^{-1}(F(A))$

  NOTE: Examples exist where $A \neq F^{-1}(F(A))$.

QUESTION: What, if any, is the relationship between the sets $F(X \backslash A)$ and $Y \backslash F(A)$?

INDUCED SET FUNCTIONS (PRE-IMAGES): Suppose $A, B \subseteq X,\ C, D \subseteq Y$ and $F : X \to Y$.

- If $C \subseteq D$, then $F^{-1}(C) \subseteq F^{-1}(D)$.

- $F^{-1}(C \cup D) = F^{-1}(C) \cup F^{-1}(D)$. More generally, $F^{-1}\left(\bigcup_{\alpha \in \Delta} C_\alpha\right) = \bigcup_{\alpha \in \Delta} F^{-1}(C_\alpha)$

- $F^{-1}(C \cap D) = F^{-1}(C) \cap F^{-1}(D)$. More generally, $F^{-1}\left(\bigcap_{\alpha \in \Delta} C_\alpha\right) = \bigcap_{\alpha \in \Delta} F^{-1}(A_\alpha)$

- $F(F^{-1}(C)) \subseteq C$

  NOTE: Examples exist where $F(F^{-1}(C)) \neq C$.

QUESTION: What, if any, is the relationship between the sets $F^{-1}(Y \setminus C)$ and $X \setminus F^{-1}(C)$?